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December 14, 2000

Keywords: 2-complexes, suspension groups, foliations, semi-flows, free group automorphisms

Abstract: *We introduce in this paper the class of* dynamical 2-complexes. *These complexes allow in particular to obtain a topological representation of any free group automorphism. A dynamical* 2*-complex can be roughly defined as a* special polyhedron *(see [17]), or* standard 2-complex *(see [5]), equipped with an orientation on its* 1*-cells satisfying two simple combinatorial properties. These orientations allow to define* non-singular semi-flows *on the complex. The relationship with the free group automorphisms is done via a cohomological criterion to* foliate *the complex by compact graphs.*

Introduction

At the origin of this work, there was the question of how to associate a notion of dynamics (symbolic or combinatorial) to a finitely presented-group. Since any such group is the fundamental group of a finite 2-complex, I was first interested in introducing a dynamic on a finite 2-complex. A wide class of 2-complexes carrying some kind of dynamic is obtained via the *suspension*, or *mapping-torus* construction. This construction consists, when given a continuous map f of a topological space X, of taking the cartesian product $X \times [0,1]$ quotiented by the equivalence relation $(x,1) \sim (f(x),0)$. When applied to a continuous map ψ of a graph Γ, this gives rise to a 2-complex, whose fundamental group admits a presentation of the form $< x_1, \cdots, x_n, t \,;\, tx_it^{-1} = \psi_\#(x_i) >$, where $\pi_1(\Gamma) \equiv < x_1, \cdots, x_n >$ and $\psi_\#\colon \pi_1(\Gamma) \to \pi_1(\Gamma)$ is an endomorphism induced by ψ. Such groups are called *mapping-torus*, or *suspension groups*. The dynamic on these suspended 2-complexes appears under the form of a *non-singular semi-flow*, whose orbits are unions of intervals $\{x\} \times [0,1]$, $x \in \Gamma$, glued together by the map ψ.

Non-singular semi-flows seemed then to be quite natural dynamical objects for my purpose. Indeed, in addition of the above remark, several classes of 2-dimensional complexes carrying non-singular semi-flows had already been studied. These were the *templates* of Williams (see [25] or [4] for instance) and the *dynamic branched surfaces* of Christy (see [6, 8]). Both have their roots in the notion of *branched surface* introduced by Williams in [24]. However, on one hand, the fundamental groups of templates are only free groups. On the other hand, because their semi-flows come from hyperbolic flows, these authors always assume the existence of a

differentiable structure on the complexes. But most of the 2-complexes constructed via the suspension operation do not admit such a smooth structure in an obvious way. Thus, it is far from clear that the class of dynamic branched surfaces allow to obtain all the mapping-torus groups. The main feature of the dynamical 2-complexes introduced in this paper is to substitute to the above condition of smoothness, not natural from an algebraic point of view, easy combinatorial properties which allow the definition of a non-singular semi-flow. In section 5.1, the reader will find a more complete discussion about the relationship between dynamical 2-complexes and the different kinds of 2-complexes presenting some dynamical aspect.
The 2-complexes considered here have their origins in a particular class of complexes introduced by Casler, Ikeda or Matveev (see [5, 16, 17]), called *standard complexes*, *closed fake surfaces* or *special polyhedra*. The difference between a standard complex and a closed fake surface is essentially the assumption, for a standard complex, to admit an embedding in some compact 3-manifold. A standard complex is a special 2-polyhedron whose complement of the set of *singular points* is a union of disjoint 2-cells, a singular point being a point where the complex is not a surface. This kind of 2-complexes was introduced by Casler for the study of compact 3-manifolds with boundary. He proved in particular that any such manifold is the "thickening" of a standard complex. Later, any finitely generated group was proved to be the fundamental group of a closed fake surface (see [26]).
Definitions and basic results about dynamical 2-complexes are stated in section 1.2. Roughly speaking, a dynamical 2-complex is combinatorially a special polyhedron, whose complement of the singular set is a union of discs, annuli and Moebius-bands, and which is moreover equipped with an orientation on its singular 1-cells, satisfying two simple combinatorial properties. This orientation allows to define a non-singular semi-flow by giving, in some sense, its direction (see section 4).
The dynamical 2-complexes stand then at the cross-roads of different branches of mathematics such as combinatorial group theory or 3-dimensional topology. Since the introduction of the dynamic on these complexes only depends of their combinatorics, a natural question is to understand what are the relationships between the combinatorics of the complex, the properties of the dynamical systems (K, σ_t), the fundamental group of the complex and the topology of the underlying manifold, if any. In particular, from what precedes, two immediate questions in mind are on one hand to know whether or not all the mapping-torus groups are the fundamental group of some dynamical 2-complex, and on the other hand to have a criterion to decide if a given dynamical 2-complex is the suspension of a graph-map.
Since a long time, the suspension, classicaly applied to a homeomorphism h of a compact manifold M^n, has been of central interest as well in topology than in dynamical systems. From a topological point of view, the $(n+1)$-manifolds M^{n+1} obtained in this way share the property to admit a non-singular, transversely orientable codimension 1-foliation with compact leaves homeomorphic to M^n. In other words, there exists a locally trivial fibration over the circle, with fiber M^n, and with monodromy the isotopy-class of h. From a dynamical point of view, this is equivalent to the existence of a non-singular flow $(\phi_t)_{t\in\mathbf{R}}$ on M^{n+1} admitting a cross-section homeomorphic to M^n, with return-map the homeomorphism h. Many authors have been interested in caracterizing flows admitting cross-sections or manifolds admitting foliations with compact leaves (see for instance [10] or [23]).
In section 3, we study a similar situation in our CW-complex context. We first adapt the notion of *foliation* to our complexes (see section 2.2). We then give a necessary and sufficient criterion, of both combinatorial and cohomological nature, for the

complex considered to admit a transversely orientable foliation by compact graphs which have all the same Euler characteristic. This Euler characteristic condition is substituted to the requirement that all the leaves are homeomorphic, which would be a too strong restriction. We so caracterize, by a finite and effective criterion depending only on the combinatorics of the complex (see remark 5.4) the existence on a given flat 2-complex of a dynamical system (K, σ_t) which is the "suspension" of a dynamical system (K', h) where K' is a graph and h a continuous map of K' (the return-map of the semi-flow σ_t on K'). We show that, when considering *regular foliations*, some semi-flow $(\sigma_t)_{t\in\mathbf{R}^+}$ transverse to the leaves of the foliation induces an *automorphism* on the fundamental group of a leaf K'. Moreover, this automorphism appears as a composition of *Whitehead moves.*

One states below weak versions of our main results.

Theorem 0.1 *A dynamical* 2*-complex* K *admits a transversely orientable regular foliation* $\mathcal{F}$ *by compact graphs which have all the same Euler characteristic, if and only if* K *admits a* positive cocycle $u \in C^1(K; \mathbf{Z})$. *The complex* K *is then homotopically equivalent to the suspension of a continuous map* ψ *of a trivalent graph* Γ, *which decomposes in Whitehead moves. In particular, the fundamental group of* K *is the suspension of an automorphism of the free group* F_n, $\pi_1(\Gamma) \equiv F_n$.

Theorem 0.2 *For any continuous map* ψ *of a trivalent graph, which admits a decomposition in Whitehead-moves, there is a dynamical* 2*-complex homotopically equivalent to the suspension of* ψ, *and which admits a positive cocycle. In particular, any group which is the mapping-torus of some free group automorphism is the fundamental group of some such dynamical* 2*-complex.*

The proof of this last theorem is constructive. Theorem 0.1 above can be considered as an analog, in our CW-complex setting, of the already mentioned result of Tischler on the existence of foliations with compact leaves on manifolds (see [23]). Further analogy with the manifold-context, and more precisely with the Thurston norm on the homology of 3-manifolds (see [21]), can be found in the fact that the whole set of non-negative cocycles form a cone (see remark 5.4), in the same way that the set of cohomology classes in $H^1(M^3; \mathbf{Z})$ form cones over some faces of a well-defined convex polyhedron. This last analogy perhaps deserves to be examined in detail.

As a corollary of theorem 0.2, one has the following result:

Theorem 0.3 *Any free group automorphism is represented by a positive cocycle of some dynamical* 2*-complex (see proposition 3.9 and corollary 3.10).*

One so obtains a class of topological 2-dimensional objects which represents all the free group automorphisms (see also [14] for the case of injective, non-surjective free group endomorphisms). In a paper in preparation, we give another proof by means of a different construction. We obtain there in particular a combinatorial construction for the suspension of any pseudo-Anosov homeomorphism of a compact surface with boundary. Finally, an other paper is to come, in which one considers more closely the problem of finding sections to semi-flows on dynamical 2-complexes.

Acknowledgements: The author would like to thank his thesis advisor J.Los for his help during the elaboration of this work. Thanks are also due to F.Beguin, C.Bonatti, J.M.Gambaudo, G.Levitt, M.Lustig, Z.Nitecki for helpful conversations at differents steps of this work. The material support of the Institut Non Lineaire de Nice was also inappreciable. Finally, the author is very grateful toward the Centre de Recerca Matemàtica and the Departament de Matemàtiques of the Universitat Autònoma de Barcelona, particularly its Dynamical Systems team, for their support during the final step of this work.

1 Flat and dynamical 2-complexes

1.1 Preliminaries

We introduce here the various objects and concepts used in the next sections. The main point is the definition of a *dynamical* 2*-complex* (see definition 1.2).
If U is an open or closed subset of a topological space X, the *boundary of* U is the set $\overline{U} - \overset{\circ}{U}$, where $\overline{U}$ (resp. $\overset{\circ}{U}$) denotes the closure (resp. the interior) of U in X. A *path* (resp. *loop*) in X is a locally injective continuous map from the interval (resp. circle) to X.
We assume that the reader is familiar with basic notions about *CW-complexes* (see for instance [18]). All our complexes are piecewise-linear and, unless otherwise stated, connected, finite and compact.
The 0-cells (resp. 1-cells) of a CW-complex are called the *vertices* (resp. *edges*) of the complex. If e is an oriented edge, then e is said to be an *incoming edge* at its *terminal vertex* $t(e)$ and an *outgoing edge* at its *initial vertex* $i(e)$. Observe that an oriented edge e can be both incoming and outgoing at a same vertex, if this edge is a loop.
A *graph* is a 1-dimensional CW-complex. A path or loop in a graph Γ is *positive* (resp. *negative*) if it is oriented such that its orientation agrees (resp. disagrees) at any point with the orientation of the edges that it intersects.
The j*-skeleton* $K^{(j)}$ of a n-dimensional CW-complex K $(0 \leq j \leq n)$ is the union of all the cells in K whose dimension is less or equal to j. Clearly $K^{(n)} = K$. The *boundary* of a n-dimensional CW-complex K is the closure in K of the union of all the $(n-1)$-cells of K which are contained in the closure of exactly one open n-cell. The points which do not belong to the boundary of K form the *interior* $\overset{\circ}{K}$ of K.
We will denote by $Con(X)$ the cone over a space X, that is the space $X \times [0,1]$, where $X \times \{1\}$ is identified to a single point. Finally, we denote by Δ^3 the closed 3-dimensional simplex.
Let Γ be a graph and let $\psi\colon \Gamma \to \Gamma$ be a continuous map. We denote by $Susp_\psi(\Gamma)$ the 2-complex $\Gamma \times [0,1]/((x,1) \sim (\psi(x),0))$. This 2-complex is called the *suspension*, or *mapping-torus*, of the map ψ of Γ.
If $F_n =< x_1, \cdots, x_n >$ is the fundamental group of Γ, the fundamental group of the mapping-torus $Susp_\psi(\Gamma)$ admits a presentation of the form $< x_1, \cdots, x_n, t \; ; \; t x_i t^{-1} = \psi_\#(x_i), i = 1, \cdots, n >$, where $\psi_\#\colon \pi_1(\Gamma) \to \pi_1(\Gamma)$ is an endomorphism induced by ψ on $\pi_1(\Gamma)$. One says that the group $\pi_1(Susp_\psi(K))$ is the *suspension*, or *mapping-torus, of the endomorphism* $\mathcal{O}$ *of* $\pi_1(K)$. In the case where $\psi_\#$ is an automorphism, $\pi_1(Susp_\psi(\Gamma))$ is the semi-direct product of F_n with $\mathbf{Z}$ over $\psi_\#$.

1.2 Basic definitions

We first recall the notion of *standard complex* introduced by Casler (see [5] and also [17, 1, 20, 26]), and we introduce the derived notion of *flat* 2*-complex*.

Following [17], we call *special* 2*-polyhedron* a piecewise-linear 2-complex satisfying the following property: For any point $x \in K$, there is a neighborhood $N(x)$ of x in K, a neighborhood $N(y)$ of a point y in the interior of $Con((\partial\Delta^3)^{(1)})$, and a homeomorphism $h_x : N(x) \to N(y)$ such that $h_x(x) = y$. If one allows the point y to belong to the boundary of $Con((\partial\Delta^3)^{(1)})$, one obtains a notion of *special* 2*-polyhedron with boundary.* The points x which are in the boundary of K are those such that $h_x(x)$ is in the boundary of $Con((\partial\Delta^3)^{(1)})$. Unless otherwise stated, the complexes considered have no boundary.
Let K be a special 2-polyhedron, possibly with boundary. The *singular graph* $K^{(1)}_{sing}$ is the closure in K of the set of points x whose image under h_x belongs to a 1-cell of the interior of $Con((\partial\Delta^3)^{(1)})$. The set of *crossings* $K^{(0)}_{sing}$ is the set of points x of K such that $h_x(x)$ is the base of $Con((\partial\Delta^3)^{(1)})$. We set $K^{(2)}_{sing} = K$. The connected components of $K^{(m+1)}_{sing} - K^{(m)}_{sing}$, $0 \leq m \leq 1$, are called the $(m+1)$*-components* of the complex (the 0-components are the crossings).
With this terminology, a *standard* 2-complex, as defined by Casler, is a special 2-polyhedron whose all 2-components are 2-cells. We will call *flat* 2-complex a special 2-polyhedron whose 2-components are either 2-cells, annuli or Moebius-bands.

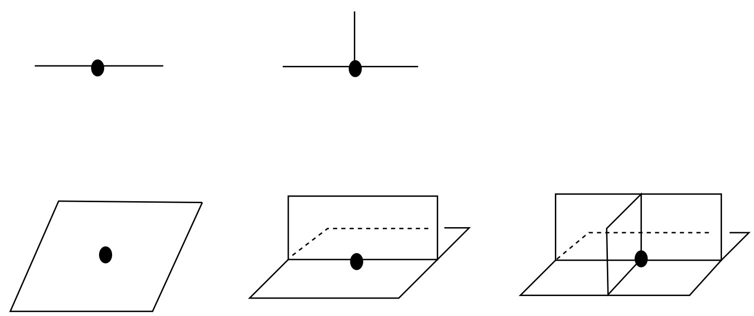


Figure 1: Non-singular and singular points in trivalent graphs and flat 2-complexes

In the following lemma, we gather some easy observations on flat 2-complexes.

Lemma 1.1 *Let K be a flat 2-complex, possibly with boundary.*

1. *Let x be any point in K, distinct from a crossing. A neighborhood of x in K is homeomorphic to a cartesian product of a triod with the interval if x is singular, and to a cartesian product of two intervals otherwise (see figure 1).*

2. *There are 4 germs of 1-components in K incident to each crossing and 6 germs of 2-components. There are 3 germs of 2-components incident to each 1-component of K. Any germ of 2-component at a crossing v contains exactly two germs of 1-components of K at v. Any pair of germs of 1-components at v is contained in exactly one germ of 2-component.*

3. *In particular, the singular graph of a flat 2-complex is a, possibly not connected, compact 4-valent graph, i.e. with 4 edges incident to each crossing.*

Important: Let K be any flat 2-complex. Then K admits a canonical structure of CW-complex defined as follows: The vertices are the crossings of the complex, together with a set of valency 2-vertices, one for each connected component of $K^{(1)}_{sing}$ which is a loop without any crossing. The edges are the 1-components of the complex, together with a set of valency 2 edges, one in each 2-component which is not *a disc. We will always assume that our flat 2-complexes K are equipped with this canonical structure of CW-complex, and their singular graph $K^{(1)}_{sing}$ with the induced structure. In particular, the edges of $K^{(1)}_{sing}$ are the 1-components of K. This causes no loss of generality for our purpose.*

Let K be a flat 2-complex, together with an orientation on the edges of the singular graph. Let C be any 2-component or 2-cell of K. We will say that C contains an *attractor* (resp. a *repellor*) in its boundary if there is a crossing v of K and a germ $g_v(C)$ of C at v such that the two germs of edges of $K^{(1)}_{sing}$ at v contained in $g_v(C)$ (see lemma 1.1, item (2)) are incoming (resp. outgoing) at v. We will say that the crossing v above *is* or *gives rise to* an attractor (resp. a repellor) for C (and for the given orientation). Observe that a same crossing v can give rise to $k > 1$ attractors or repellors in the boundary of a same component or cell C.

All the tools needed for the definition of a *dynamical* 2*-complex* have been given.

Definition 1.2 A *flat dynamical* 2*-complex* is a flat 2-complex K together with an orientation on the edges of the singular graph $K^{(1)}_{sing}$ satisfying the following two properties:

1. Each crossing of K is the initial crossing of exactly 2 edges of $K^{(1)}_{sing}$.
2. Any 2-component which is a 2-cell has exactly one attractor and one repellor for this orientation in its boundary. The other components have no attractor and no repellor in their boundary.

A *standard dynamical* 2*-complex* is a flat dynamical 2-complex which is also a standard 2-complex.

The following lemma is easily deduced from the definition of a dynamical 2-complex.

Lemma 1.3 *Let K be a flat dynamical* 2*-complex. Then the boundary circles of the annuli and Moebius-band components are positive loops in $K^{(1)}_{sing}$. The boundary circle of a disc component D decomposes as pq^{-1} where p and q are two positive paths in $K^{(1)}_{sing}$ of initial point the repellor of D and of terminal point its attractor.*

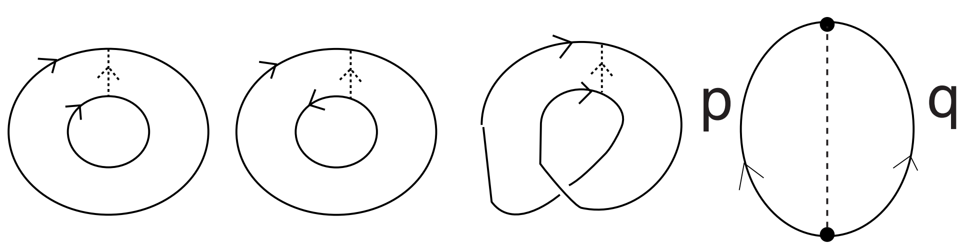


Figure 2: 2-components in a dynamical 2-complex

Remark 1.4 By definition of a flat dynamical 2-complex, the boundary circles of the annulus and Moebius-band components inherit an orientation from the orientation of the edges of the singular graph. If these orientations of the two boundary circles of a same annulus component agree, then this annulus component is called a *coherent annulus component*. Otherwise, it is an *incoherent annulus component*.

A crossing in a flat 2-complex which is the terminal crossing of exactly j edges, $0 \leq j \leq 4$, will be called a *type j-crossing.* In a flat dynamical 2-complex, by definition, there are only type 2-crossings.

Remark 1.5 Let K be a flat 2-complex and let v be a type j-crossing. Then v gives rise to 1 attractor and 1 repellor in the set of the boundaries of the 2-components if $j = 2$, 0 attractor (resp. repellor) if $j < 2$ (resp. $j > 2$) and 3 repellors (resp. attractors) if $j = 1$ (resp. $j = 3$) or 6 repellors (resp. attractors) if $j = 0$ (resp. $j = 4$). This comes from lemma 1.1, item (2).

It is not true that any flat 2-complex admits an orientation which makes it a dynamical 2-complex. The following proposition gives a property satisfied by any flat dynamical 2-complex.

Proposition 1.6 *The Euler characteristic of a flat dynamical* 2*-complex is zero.*

Proof of proposition 1.6: Let K be a flat dynamical 2-complex. Let X_1 be the number of crossings of K, X_2 the number of 1-components whose closure in K contains a crossing and X_3 the number of disc components. Then, the Euler characteristic $\chi(K)$ is equal to $X_1 - X_2 + X_3$. By lemma 1.1, the singular graph is 4-valent. Thus, $X_2 = 2 * X_1$. Therefore $\chi(K) = X_3 - X_1$. Furthermore, by item (1) of definition 1.2 and remark 1.5, each crossing of K gives rise to exactly one attractor and one repellor in the set of boundaries of disc components. Item (2) of definition 1.2 implies then that $X_1 = X_3$. This completes the proof of proposition 1.6. $\diamondsuit$

2 Cocycles, embeddings and foliations

2.1 Homology of flat complexes

We assume that the reader is familiar with the homology of CW-complexes (see for instance [18]). We just recall and introduce here some notions which will be needed later.

Let us first remind that the singular graph $K^{(1)}_{sing}$ of a complex K is always assumed to be equipped with a structure of CW-complex whose 0-cells are the crossings of K, together with a set of valency 2-vertices in the loops containing no crossing, and whose 1-cells are the 1-components of K. Furthermore, these edges and vertices of $K^{(1)}_{sing}$ are the only edges and vertices of K contained in $K^{(1)}_{sing}$.

Let K be a flat 2-complex. An *integer cocycle* will denote a cocycle in $C^1(K; \mathbf{Z})$, that is a finite collection of integer weights on the edges of the complex whose algebraic sum along the boundary circles of the 2-cells of K is zero.

The following definitions are stated for any flat 2-complex, but they will be considered only in the case of dynamical 2-complexes. Let us recall that, in this case, the edges of the singular graph of the complex are equipped with a well-defined orientation (see definition 1.2).

Definition 2.1 Let K be a flat 2-complex, together with an orientation on the edges of its singular graph.

1. A *non-negative* cocycle in $C^1(K; \mathbf{Z})$ is an integer cocycle u such that $u(e) \geq 0$ holds for any edge e in the singular graph $K^{(1)}_{sing}$ and there is at least one such edge e with $u(e) > 0$.

2. A *positive* cocycle is a non-negative cocycle which is positive on all the positive embedded loops in $K^{(1)}_{sing}$.

The δ_v*-moves* defined below consist of removing 1 from the value of an integer cocycle u on each incoming edge at some crossing v of a flat 2-complex K, and adding 1 to its value on the outgoing one. More precisely:

Definition 2.2 Let K be a flat 2-complex. Let $u \in C^1(K; \mathbf{Z})$ be a cocycle and let v be any crossing of K.
A δ_v*-move* is the map $\delta_v: C^1(K; \mathbf{Z}) \to C^1(K; \mathbf{Z})$ defined by:
$(\delta_v(u))(e_i) = u(e_i) - 1$ for all the incoming, non-outgoing 1-cells e_i of K at v,
$(\delta_v(u))(f_j) = u(f_j) + 1$, for all the outgoing, non-incoming 1-cells f_j of K at v,
$(\delta_v(u))(x) = u(x)$ for all the other 1-cells x in $K^{(1)}$.

Remark 2.3 The image of a cocycle by a δ_v-move is a cocycle. Furthermore, two integer cocycles u' and u of a flat 2-complex K which are obtained one from the other by a finite sequence of δ_v-moves are cohomologous.

Definition 2.4 With the assumptions and notations of definition 2.2,
A *non-negative* δ_v*-move* on a non-negative cocycle u is a δ_v-move on u such that $\delta_v(u)$ is non-negative.

Remark 2.5 We call *non-negative* (resp. *positive*) a cohomology class $c \in H^1(K; \mathbf{Z})$ such that $c(l) \geq 0$ (resp. $c(l) > 0$) holds for any positive embedded loop l in $K^{(1)}_{sing}$. Any non-negative integer cocycle defines a non-negative cohomology class. The converse is true, that is: *Let K be a flat dynamical 2-complex. Any non-negative cohomology class in $H^1(K; \mathbf{Z})$ is represented by a non-negative cocycle in $C^1(K; \mathbf{Z})$.* Since this is not essential here, and the proof is rather long and combinatorial, we postpone this proof to another paper to come, where this result appears to be of greater importance.

2.2 Embeddings and Foliations

We will say that a graph Γ_1 embedded in a flat 2-complex K is *c-transverse* to an embedded graph Γ_2 if, for any x in $\Gamma_1 \cap \Gamma_2$, for any small neighborhood $N(x)$ of x in K, for any isotopy j^1_t (resp. j^2_t), $t \in [0, 1]$, of Γ_1 (resp. of Γ_2) in K with support in $N(x)$, $j^1_t(\Gamma_1) \cap j^2_t(\Gamma_2)$ is non-empty. If Γ_1 and Γ_2 are not c-transverse at some point $x \in \Gamma_1 \cap \Gamma_2$, they are *c-tangent* (at x).

By definition of a flat 2-complex K, for any 2-component C of K, there is a compact surface with boundary $\overline{S}_C$, and a continuous *attaching-map* $h_C: \overline{S}_C \to \overline{C}$, which is a homeomorphism from its interior onto C and which sends its boundary to the boundary of C in K. In particular, S_C is either a disc, an annulus or a Moebius-band according to the homeomorphism type of C.

Definition 2.6 A *regular foliation* $\mathcal{F}$ of a flat 2-complex K is a decomposition of K in disjointly embedded, possibly non finite, graphs, called the *leaves* of $\mathcal{F}$, satisfying the following properties:

1. If C is any annulus or Moebius-band component, then $h_C^{-1}(\mathcal{F} \cap \overline{C})$ is a non-singular foliation of $\overline{S}_C$ by intervals transverse to the boundary.

2. If C is any disc component, then $h_C^{-1}(\mathcal{F} \cap \overline{C})$ is a foliation of $\overline{S}_C$ with exactly two leaves reduced to two points in the boundary of $\overline{S}_C$, and which is transverse to the boundary of $\overline{S}_C$ outside these two points. Furthermore:

    (a) The images of these two points under h_C are one or two crossings of K.

    (b) The foliation $h_C^{-1}(\mathcal{F} \cap C)$ is a non-singular foliation by lines of S.

Let us consider any leaf of a regular foliation of a flat 2-complex K. This leaf is the image in K under a particular kind of embedding of a possibly non-finite graph. This kind of embeddings of a graph will be called *r-embeddings*. If the graph contains a crossing of the complex, then the r-embedding is called *degenerate*. The following lemma is a direct consequence of this definition, and caracterizes the r-embeddings.

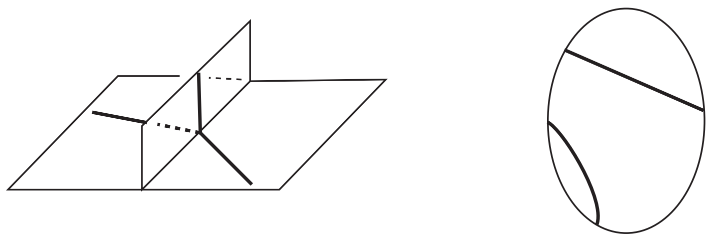


Figure 3: A r-embedding

Lemma 2.7 *Let Γ be a graph which is r-embedded in a flat 2-complex K.*

1. *The crossings of Γ belong to the singular graph $K^{(1)}_{sing}$. Any intersection point of Γ with an open 1-component of K is a crossing of Γ.*

2. *Each crossing x of Γ interior to an edge e of $K^{(1)}_{sing}$ has exactly three incident germs of edges of Γ, one in each germ of 2-component of K incident to e.*

3. *If v is a crossing of K contained in Γ, and if $N_\Gamma(v)$ denotes a neighborhood of v in Γ, then $N_\Gamma(v) - v$ has at most six connected components, at most one in each germ of 2-component of K at v.*

Conversely, any graph embedded in K and satisfying the properties above is the image of a r-embedding. In particular, a graph which is r-embedded in K is c-transverse to the singular graph of K.

Let K be a flat 2-complex. We need in the sequel of the papers the notions of *being 2-sided*, *transversely orientable* and *transversely oriented* for a graph r-embedded in K, or, with respect to the two last notions, for a regular foliation of K. All the definitions are adapted in a more or less straightforward way from the same usual definitions for surfaces embedded in compact 3-manifolds, and non-singular foliations of such manifolds, and moreover agree with the intuition. For the sake of brievety, we thus leave these small adaptations to the reader. However, it may be worthwhile being aware of remark 2.8 below.

Remark 2.8 Let K be a flat 2-complex and let p be a r-embedding of a graph Γ in K. Assume that Γ is compact. Then $p(\Gamma)$ is 2-sided in K if and only if there is a neighborhood $\mathcal{N}(p(\Gamma))$ of $p(\Gamma)$ in K such that:

$\mathcal{N}(p(\Gamma))-p(\Gamma)$ has two connected components C_- and C_+ which are trivial I-bundles respectively over Γ_- and Γ_+, where Γ_- and Γ_+ are two graphs r-embedded in K (in a non-degenerate way).
If p is a non-degenerate r-embedding, being 2-sided is equivalent to require that there is a neighborhood of $p(\Gamma)$ in K which is the trivial I-bundle over $p(\Gamma)$. One thus has the usual notion of being 2-sided for a surface in a 3-manifold.

In what follows, we will often omit the embedding map p and speak of a *graph* Γ *embedded in* K.

Lemma 2.9 *Let K be a flat 2-complex. Any cocycle $u \in C^1(K;\mathbf{Z})$ defines a transversely oriented graph K_u which is r-embedded in K. Furthermore, at each point $x \in K_u \cap e$, where e is any 1-cell of K, the transverse orientation to K_u agrees with the orientation of e if $u(e) > 0$ and disagrees otherwise.*
Conversely, any transversely oriented graph K_u r-embedded in K in a non-degenerate way defines a unique cocycle in $u \in C^1(K;\mathbf{Z})$.

Proof of lemma 2.9: Let e be any edge of the 1-skeleton of K with $u(e) = \pm k$, $k > 0$. Let $x_1, \cdots, x_k$ be k points in e, each with a weight of $+1$ or -1 according to whether $u(e) = +k$ or $u(e) = -k$. Let j be the smallest integer such that the points x_i belong to $K^{(j)}_{sing}$. By lemma 1.1, the neighborhood of any x_i is homeomorphic to $X_i \times [0,1]$, where X_i is either an interval if x_i is non-singular, or a triod otherwise. One considers now around each x_i an embedded piece $X_i \times \{t_i\}$, $t_i \in [0,1]$. One does the same thing for any edge in $K^{(1)}$. By definition of a cocycle with coefficients in $\mathbf{Z}$, the germs of 1-cells in these X_i can be connected by 1-cells embedded in the corresponding 2-cells of K. Moreover, these 1-cells are equipped with a transverse orientation which agrees at x_i with the orientation of e if the weights are positive and disagrees otherwise.
If all the 1-cells so embedded in a same 2-cell of K are not disjointly embedded, then classical cut-and-paste technics respecting the transverse orientation allow to obtain an embedded complex in K. By construction, this complex is the image in K of a r-embedding of a graph. It is also transversely oriented, the values $u(e)$ being its intersection numbers with the oriented edges e of the singular graph. This last remark allows to prove the converse assertion. $\diamondsuit$

Remark 2.10 The definitions of r-embedding and regular foliation are adapted to the case of flat 2-complexes with boundary, by imposing that a leaf of a foliation, i.e. a r-embedded graph, in such a complex is either disjoint from the boundary, or is a connected component of this boundary.

2.3 Whitehead moves and Elementary Foliated complexes

If f is a map from a graph Γ to a graph Γ', we will write $\sim_f$ the equivalence relation defined by $x \sim_f y$ if and only if $f(x) = f(y)$. We will denote by $\pi_f\colon \Gamma \to \Gamma/\sim_f$ the associated projection-map. If e is an edge of Γ, we write $\sim_e$ for the relation $x \sim_e y$ if and only if x and y are in e. By a *collapsing* C_e from a graph Γ to a graph Γ', we will mean the projection-map $\Gamma \to \Gamma/\sim_e$, with $\Gamma/\sim_e = \Gamma'$. This is a homotopy equivalence if the closure of e is not a loop. We will call *splitting* S_v at a vertex v of a graph Γ a continuous map from Γ to a graph Γ', which is an inverse, up to homotopy, of some collapsing C_e from Γ' to Γ. We will call *Whitehead-move* from a graph Γ to a graph Γ' a composition $S_{C_e(e)} \circ C_e$ where the splitting $S_{C_e(e)}$ is not a homotopy-inverse of the collapsing C_e (see figure 5). A Whitehead-move is clearly a homotopy equivalence. The motivation for their introduction is explained by lemma 2.15 below (see also proposition 3.9).

Definition 2.11 1. A *dynamic collapse* from a graph Γ to a graph Γ' is a continuous deformation $\pi_{f_t^e}$, $t \in [-1, 0]$, such that:

- For any $-1 \leq t \leq 0$, the map $f_t^e \colon \Gamma \to \Gamma$ is the identity outside a contractible neighborhood of e in Γ which does not contain any vertex distinct from $i(e)$ and $t(e)$.
- If $h_e \colon \overline{e} \to [-1, 1]$ is some homeomorphism, then $f^e_{t\,|\overline{e}}$ is equal to $h_e^{-1}(-th_e)$.
- The map $\pi_{f_0^e}$ is the collapsing $C_e \colon \Gamma \to \Gamma'$.

2. A *dynamic splitting* from a graph Γ to a graph Γ' is a continuous deformation $S_v^t \colon \Gamma \to \Gamma^t$, $t \in [0, 1]$, such that:

- $\Gamma^0 \equiv \Gamma$ and $S_v^0 \colon \Gamma \to \Gamma$ is the identity-map.
- For any $t > 0$, $\Gamma^t \equiv \Gamma'$ and S_v^t is a splitting at v from Γ to Γ'.

Definition 2.12 A *dynamic W-move* from a graph Γ to a graph Γ' is a continuous deformation $W_e^t \colon \Gamma \to \Gamma'$, $t \in [-1, 1]$ satisfying the following properties:

1. For $t \in [-1, 0]$, W_e^t is a dynamic collapse $\pi_{f_t^e} \colon \Gamma \to \Gamma / \sim_{f_t^e}$, and for $t \in [0, 1]$, W_e^t is a dynamic splitting $S^t_{\pi_{f_0^e}(e)} \colon \pi_{f_0^e}(\Gamma) \to \Gamma^t$.

2. The map $S^1_{\pi_{f_0^e}(e)} \circ \pi_{f_0^e}$ is a Whitehead-move W_e from Γ to Γ'.

Definition 2.13 Let W_e^t, $t \in [-1, 1]$, be a dynamic W-move from a graph Γ to a graph Γ'.

The 2-complex $\mathcal{C}_{\Gamma\Gamma'} = \displaystyle\bigcup_{t=-1}^{1} W_e^t(\Gamma)$ is called an *elementary foliated 2-complex*.

See figure 4 or 5.

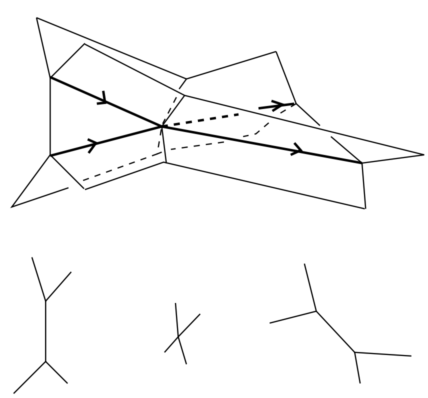


Figure 4: A Whitehead-move I

These elementary foliated complexes will be the basic pieces in the construction of a transversely orientable regular foliation of a dynamical 2-complex, starting from a positive cocycle of the complex (see lemma 2.15 and proposition 3.8). They are also used in proposition 3.9.

Lemma 2.14 *Let W^t_{e0} be a dynamic W-move from a trivalent graph Γ_0 to a trivalent graph Γ_1. Then an elementary foliated 2-complex $\mathcal{C}_{\Gamma_0\Gamma_1}$ is a standard 2-complex with two boundary components $\Gamma_0 \times \{-1\}$ and $\Gamma_1 \times \{1\}$ which admits a transversely orientable regular foliation with compact leaves $\mathcal{F}$. Moreover, all the non-degenerate leaves of $\mathcal{F}$ are homeomorphic either to Γ_0 or Γ_1. There is exactly one degenerate leaf, containing the only crossing of $\mathcal{C}_{\Gamma_0\Gamma_1}$, which is homeomorphic to the graph $C_{e_0}(\Gamma_0)$.*

See figure 4 or 5.

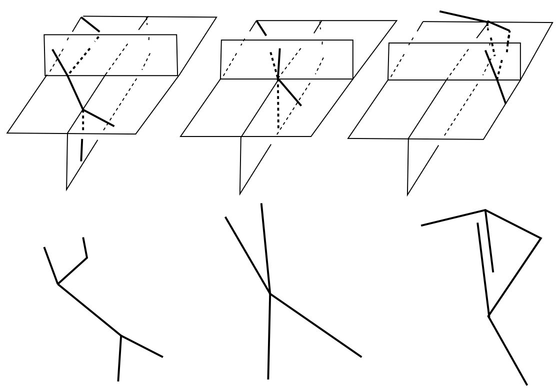


Figure 5: A Whitehead move II

Proof of lemma 2.14: From definition 2.13, $\mathcal{C}_{\Gamma_0\Gamma_1}$ is parametrized by coordinates (x,t) ($t \in [-1,1]$). It is clear that $\mathcal{C}_{\Gamma_0\Gamma_1}$ is a 2-dimensional CW-complex with two boundary components $\Gamma_0 \times \{-1\}$ and $\Gamma_1 \times \{1\}$. For $t < 0$ (resp. $t > 0$), the level sets are graphs homeomorphic to Γ_0 (resp. Γ_1). The level set $t = 0$ is the graph $C_{e_0}(\Gamma_0)$. From definition 2.11, any point x in $\mathcal{C}_{\Gamma_0\Gamma_1}$, distinct from $C_{e_0}(e_0)$, admits a neighborhood of the form $N_{\Gamma_i}(x)$ crossed with the interval, where $N_{\Gamma_i}(x)$ is a neighborhood of x in Γ_i ($i \in \{1,2\}$). Since the graphs Γ_i are trivalent, $N_{\Gamma_i}(x)$ is either a triod or an interval. Therefore, all the points of an elementary foliated complex, which are distinct from $C_{e_0}(e_0)$, satisfy the definition of a special 2-polyhedron. By definition of collapsings and splittings, the point $C_{e_0}(e_0)$ is a cone over a 1-cell in $\Gamma_0 \times \{-1\}$ and also of an other in $\Gamma_1 \times \{1\}$. The continuity of a dynamic W-move implies that the four germs of edges of Γ_0 incident to the vertices of e_0 are simply translated by the dynamic W-move to edges of $C_{e_0}(\Gamma_0)$, incident to the vertex $C_{e_0}(e_0)$, and then translated to four germs of edges of Γ_1 incident to the vertices of the splitted edge. This implies that $C_{e_0}(e_0)$ is a singular point and, in a neighborhood of $C_{e_0}(e_0)$, the set of singular points is a cross X with four edges. By definition, the map W^1_{e0} is a Whitehead-move from Γ_0 to Γ_1. Therefore, it is not homotopic to the identity-map of Γ_0. This implies that any pair of edges of the above cross X is contained in exactly one germ of 2-cell. Therefore, from which precedes, some homeomorphism takes a neighborhood of $C_{e_0}(e_0)$ in K to the cone over the 1-skeleton of the boundary of the 3-simplex, the image of $C_{e_0}(e_0)$ under this homeomorphism being the base of this cone. Since the complement of the set of singular points is clearly a union of discs, one so proved that an elementary foliated complex is a special 2-polyhedron. All the other assertions of lemma 2.14 follow easily from the construction and the above observations. $\diamondsuit$

Lemma 2.15 below allows to understand the relationship between a non-negative δ_v-move on a non-negative cocycle u in a dynamical 2-complex K and a Whitehead move.

Lemma 2.15 *Let K be a flat dynamical 2-complex which admits a non-negative cocycle $u \in C^1(K;\mathbf{Z})$. If v is a crossing of K such that $\delta_v(u)$ is a non-negative cocycle, let Γ_u and $\Gamma_{\delta_v(u)}$ be as given by lemma 2.9.*
Then there is a subcomplex of K containing v which is homeomorphic to an elementary foliated complex $\mathcal{C}_{\Gamma_u \Gamma_{\delta_v(u)}}$. In particular, the graph $\Gamma_{\delta_v(u)}$ is obtained from the graph Γ_u by a Whitehead move.

Proof of lemma 2.15: By lemma 2.9, any r-embedded graph Γ_u is 2-sided in K, and equipped with a transverse orientation which agrees with the orientation of the edges of the singular graph that it intersects. Since $\delta_v(u)$ is a non-negative cocycle, the cocycle u is positive on the incoming edges of $K^{(1)}_{sing}$ at v. Thus, some germ of the 2-component C of K containing v as attractor in its boundary can be assumed to contain a 1-component e_0 of Γ_u. Clearly, e_0 cuts C in two connected components. From the definition of a special 2-polyhedron, one of these two connected components is a cone over e_0 based at the crossing v.
Since Γ_u is 2-sided, one can then define a continuous deformation

$$H_0\colon \begin{array}{ccccc} \Gamma_u & \times & [-1,0] & \to & K \\ (x & , & t) & \to & i_t(x) \end{array} \quad \text{such that:}$$

- For $t \in [-1,0[$, $i_t\colon \Gamma_u \to K$ is a non-degenerate r-embedding.
- The map i_0 is such that $i_0(\Gamma_u)$ contains v and is the image under a degenerate r-embedding of the graph Γ obtained from Γ_u by the collapsing of e_0.
- For any t, t' in $[-1,0]$ such that $t \neq t'$, $i_t(\Gamma_u)$ is disjoint from $i_{t'}(\Gamma_u)$.

One denotes by $\mathcal{C}_{\Gamma_u}$ the subcomplex of K equal to $\displaystyle\bigcup_{(x,t)\in\Gamma_u\times[-1,0]} H_0(x,t)$.

The r-embedded graph Γ is 2-sided in K. It admits a neighborhood $N(\Gamma)$ in K such that $N(\Gamma) - \Gamma$ has two connected components homeomorphic to $\Gamma_u \times [-1,0[$ and $\Gamma_{\delta_v(u)} \times]0,1]$. Thus there is a r-embedding of $\Gamma_{\delta_v(u)}$ which is disjoint from $\mathcal{C}_{\Gamma_u}$. Moreover one can choose this r-embedding of $\Gamma_{\delta_v(u)}$ such that the neighborhood $N(v)$ above contains the closure of a 1-component e_1 of $\Gamma_{\delta_v(u)}$. This component e_1 is contained in the component of K which has v as repellor in its boundary. In the same way than above, one then defines a continuous deformation $H_1(x,t)$ of $\Gamma_{\delta_v(u)}$ in K. One denotes by $\mathcal{C}_{\Gamma_{\delta_v(u)}}$ the subcomplex of K equal to $\displaystyle\bigcup_{(x,t)\in\Gamma_{\delta_v(u)}\times[-1,0]} H_1(x,t)$.

By construction, the subcomplex $\mathcal{C}_{\Gamma_u} \cup \mathcal{C}_{\Gamma_{\delta_v(u)}}$ is homeomorphic, by a "fiber-preserving" homeomorphism, to the elementary foliated complex $\mathcal{C}_{\Gamma_u \Gamma_{\delta_v(u)}}$. Lemma 2.15 follows. $\diamondsuit$

3 Foliations with compacts leaves

In this section, we prove our main theorem.

Theorem 3.1 *A flat 2-complex K admits a transversely orientable regular foliation $\mathcal{F}$ by compact graphs which have all the same Euler characteristic, if and only if there exists an orientation of the edges of the singular graph such that:*

1. *The complex K together with this orientation is a flat dynamical 2-complex.*

2. *There is a positive cocycle in $C^1(K;\mathbf{Z})$, for K equipped with this orientation.*

If $\mathcal{L}$ is any leaf of $\mathcal{F}$, there is a homotopy equivalence $\psi: \mathcal{L} \to \mathcal{L}$ (which in particular induces an automorphism on the fundamental group of $\mathcal{L}$) such that the complex K is homotopically equivalent to the suspension $Susp_\psi(\mathcal{L})$.

Let us recall that a positive cocycle of K is a cocycle which is non-negative on the edges of the singular graph $K^{(1)}_{sing}$, and positive on all the positive loops embedded in $K^{(1)}_{sing}$. Let us also recall that, by edges of the singular graph, we mean the 1-components of the complex K. These are the only 1-cells of K contained in $K^{(1)}_{sing}$. The only 0-cells of K contained in $K^{(1)}_{sing}$ are the crossings, together with a set of valency 2-vertices, one in each loop of $K^{(1)}_{sing}$ which does not contain any crossing. All this comes from the chosen structures of CW-complex for our complexes.

3.1 From a regular foliation to a positive cocycle

We assume here that one is given a regular foliation $\mathcal{F}$ of a flat 2-complex K as in theorem 3.1. One first proves that there is an orientation on the edges of the singular graph, induced by some transverse orientation to $\mathcal{F}$, which makes K a flat dynamical 2-complex. The proof of the existence of a positive cocycle will then be an easy task.

Proposition 3.2 *Let K be a flat 2-complex. If there exists a transversely orientable regular foliation $\mathcal{F}$ of K by compact graphs which have the same Euler characteristic, then there is an orientation of the edges of the singular graph of K such that K together with this orientation is a flat dynamical 2-complex.*

Proof of proposition 3.2: By lemma 2.7, the leaves of a regular foliation of a flat 2-complex are c-transverse to the singular graph (see definition 2.6 and lemma 2.7). Thus, any transverse orientation of such a foliation $\mathcal{F}$ induces an orientation on the edges of the singular graph. One has to check that such an orientation satisfies the two properties of definition 1.2. We are first going to prove that property (1) is satisfied. This will essentially relie on the fact that all the leaves of $\mathcal{F}$ have the same Euler characteristic.

Lemma 3.3 *With the assumptions and notations of proposition 3.2, assume that the edges of the singular graph of K are equipped with some transverse orientation to $\mathcal{F}$.*
Let $\mathcal{L}$ be a degenerate leaf of $\mathcal{F}$, and let $\{v_1, \cdots, v_r\}$ be the crossings of K contained in $\mathcal{L}$. Then, there are some small neighborhoods $\mathcal{N}(v_i)$ of the v_i in K such that, for any $i \in \{1, \cdots, r\}$, each germ of 2-component in $\mathcal{N}(v_i)$, which contains only germs of incoming edges, or only germs of outgoing edges, of $K^{(1)}_{sing}$ at v_i, contains exactly one closed 1-component of any leaf $\mathcal{L}'$ in a sufficiently small neighborhood of $\mathcal{L}$ in K. These are the only germs of 2-components at the v_i satisfying this property. The components of $\mathcal{L}'$ not contained in these neighborhoods are in bijection with the components of $\mathcal{L}$ distinct from the v_i.

Proof of lemma 3.3: By definition of a transversely orientable foliation, each leaf is 2-sided in K. By definition of a 2-sided embedding in a flat 2-complex (see remark 2.8), there is a neighborhood $\mathcal{N}(\mathcal{L})$ of $\mathcal{L}$ in K such that $\mathcal{N}(\mathcal{L})-\mathcal{L}$ has two connected components which are homeomorphic to $\mathcal{L}_- \times [-1,0[$ and $\mathcal{L}_+ \times]0,1]$, where $\mathcal{L}_- \times \{-t\}$ and $\mathcal{L}_+ \times \{t\}$ are r-embedded in K for any $t \in]0,1]$.
All the leaves $\mathcal{L}_- \times \{-t\}$ (resp. $\mathcal{L}_+ \times \{t\}$) have crossings along the incoming (resp. outgoing) edges of $K^{(1)}_{sing}$ at the crossings v_i of K. These crossings are ordered along the edges of the singular graph containing them. One chooses $\epsilon > 0$ sufficiently small and a small neighborhood in K of each v_i so that only the last (resp. first) crossings of $\mathcal{L}_1 = \mathcal{L}_- \times \{-\epsilon\}$ (resp. $\mathcal{L}_2 = \mathcal{L}_+ \times \{\epsilon\}$) along the incoming (resp. outgoing) edges at each crossing v_i are contained in this small neighborhood.
By definition of the orientation of the edges of the singular graph, the only 1-components of $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) which might be contained in these neighborhoods of the crossing v_i are the one intersecting the germs at v_i of 2-components in K which contain only incoming (resp. outgoing) germs of edges at v_i. Since $\mathcal{F}$ is transversely orientable, and transversely oriented by the edges of $K^{(1)}_{sing}$, any such germ at v_i of any 2-component in K contains a 1-component of $\mathcal{L}_1$ (resp. $\mathcal{L}_2$), if the ϵ above is chosen sufficiently small. This proves lemma 3.3. $\diamondsuit$

One can now prove the following

Lemma 3.4 *With the notations and assumptions of proposition 3.2, if the edges of the singular graph of K are equipped with the orientation induced by any transverse orientation to $\mathcal{F}$, each crossing of K is the initial crossing of exactly two such edges.*

Proof of lemma 3.4: From lemma 3.3, if there are type 0- or type 4-crossings, all the germs of 2-components at these crossings contain 1-components of leaves of $\mathcal{F}$. This implies that these crossings are leaves of the foliation, which is impossible.
Assume now that there is a type 1-crossing v. Let us recall that the Euler characteristic of a graph is the alternated sum of the number of its 0- and 1-components. By remark 1.5, v gives rise to 3 repellors in the set of the boundaries of the 2-components. Consider the leaf $\mathcal{L}$ of $\mathcal{F}$ which contains this crossing v. If v is the only crossing contained in $\mathcal{L}$, then, by lemma 3.3, there is a leaf $\mathcal{L}_+$ in a neighborhood of $\mathcal{L}$ and a neighborhood $\mathcal{N}(v)$ of v in K which contains three 1-components and three crossings of $\mathcal{L}_+$ whereas all its other 1-components and crossings are in bijection with the 1-components and crossings of $\mathcal{L}$ distinct from v. This implies $\chi(\mathcal{L}) > \chi(\mathcal{L}_+)$, which contradicts our assumption. Assume now that $\mathcal{L}$ contains another crossing w of K. Using lemma 3.3 and remark 1.5, a simple calculation allows to prove that, whatever type has the crossing w, the alternated sum of the number of crossings and 1-components of $\mathcal{L}_+$ contained in a well-chosen neighborhood $\mathcal{N}(w)$ is less or equal to 1, which is the alternated sum of the number of crossings and 1-components of $\mathcal{L}$ contained in $N(w)$ (respectively one and zero). The crossings and 1-components of $\mathcal{L}$ and $\mathcal{L}_+$ not contained in $\mathcal{N}(w) \cup \mathcal{N}(v)$ are in bijection. Therefore, $\chi(\mathcal{L}) > \chi(\mathcal{L}_+)$. This proves that there does not exist type 1-crossings. The non-existence of type 3-crossings is proved in the same way. $\diamondsuit$

For proving that K, equipped with the above orientation, is a flat dynamical 2-complex, it remains to prove that there is exactly one attractor and one repellor in the boundary of each 2-component which is a disc and no attractor nor repellor in the boundary of the other 2-components. On one hand, each attractor or repellor in the boundary of a component C is, by definition of the transverse orientation to $\mathcal{F}$, a leaf of $h_C^{-1}(\mathcal{F} \cap \overline{C})$ which is reduced to a single point (h_C is the attaching-map of the component C, see section 2.2). On the other hand, definition 2.6 implies that there are no such leaves in the boundary of $h_C^{-1}(\overline{C})$ if C is an annulus or a Moebius-band and there are exactly two if C is a disc component. The transversal orientability to $\mathcal{F}$ implies that neither two attractors nor two repellors exist in the boundary of a same disc component. All these assertions together complete the proof of proposition 3.2. $\diamondsuit$

The existence of a positive cocycle for K, equipped with an orientation on the edges of $K^{(1)}_{sing}$ as in lemma 3.4, is now straightforward. The union of all the non-degenerate leaves of the foliation $\mathcal{F}$ intersects all the positive embedded loops of the singular graph $K^{(1)}_{sing}$. Since $K^{(1)}_{sing}$ is finite, there is a finite family of leaves of $\mathcal{F}$ whose union intersects all the positive embedded loops of $K^{(1)}_{sing}$. By lemma 2.9, the integer cocycle associated to this union of leaves is positive. The proof of one implication of theorem 3.1 is thus completed.

Remark 3.5 *Let K be a flat 2-complex which admits a transversely orientable regular foliation with compact leaves. Then all the leaves of $\mathcal{F}$ are homotopic in K.* This is easily deduced from the 2-sidedness of the leaves and the compacity of K. The cocycles associated to the non-degenerate leaves chosen above are therefore cohomologous. Thus, *each* leaf in this finite family intersects all the positive loops of $K^{(1)}_{sing}$, and so defines a positive cocycle.

3.2 The reverse implication

The goal now is to prove that, if a flat dynamical 2-complex admits a positive cocycle $u \in C^1(K;\mathbf{Z})$, then there is a transversely oriented regular foliation of K, whose leaves are compact and have the same Euler characteristic, and such that their transverse orientation agrees with the orientation of the edges of the singular graph.

Lemma 3.6 *Let K be a flat dynamical 2-complex which admits a positive cocycle $u \in C^1(K;\mathbf{Z})$. Then either K has no crossing, or there is a crossing v of K such that $\delta_v(u)$ is a positive cocycle.*

The assumption that K is a dynamical 2-complex allows to assure, in the proofs of lemma 3.6 and corollary 3.7, that there is at least one incoming edge of $K^{(1)}_{sing}$ at each crossing of K.

Proof of lemma 3.6: This lemma relies on the following claim, which is easy to check from the definition of a δ_v-move (see definitions 2.2 and 2.4).

Claim: Let $u \in C^1(K;\mathbf{Z})$ be a non-negative cocycle and let v be any crossing of K. A non-negative δ_v-move can be applied to u if and only if $u(e) > 0$ for all the incoming edges e of $K^{(1)}_{sing}$ at v.

Assume that K has at least one crossing and that a crossing as given by lemma 3.6 does not exist. Equivalently, by the claim above, there is at least one incoming edge e of $K^{(1)}_{sing}$ at each crossing v of K such that $u(e) = 0$. Let v_1 be any crossing and let e_{i_1} be an incoming edge at v_1, with $u(e_{i_1}) = 0$. There is also, by assumption, an incoming edge e_{i_2} of $K^{(1)}_{sing}$ at the initial crossing $i(e_{i_1})$ with $u(e_{i_2}) = 0$. The path $e_{i_2}e_{i_1}$ is a positive path in the singular graph $K^{(1)}_{sing}$, and u is zero on this path. By induction, from the finiteness of $K^{(1)}_{sing}$, one so constructs a positive loop l in $K^{(1)}_{sing}$ with $u(l) = 0$. This is a contradiction with u being a positive cocycle and completes the proof of lemma 3.6. $\diamondsuit$

Corollary 3.7 *With the assumptions and notations of lemma 3.6, if K has at least one crossing, then there is a (non-unique) ordered sequence of N distinct positive cocycles $u = u_0 \to u_1 \to \cdots \to u_N = u$ such that u_i is obtained from u_{i-1} by a δ_{v_i}-move, $i = 1, \cdots, N$. Furthermore, each crossing of K occurs exactly once in the set $\{v_1, \cdots, v_N\}$.*

Proof of corollary 3.7: By lemma 3.6, if $u \in C^1(K;\mathbf{Z})$ is a positive cocycle, there is at least one crossing v_1 of K such that $u_1 = \delta_{v_1}(u)$ is a positive cocycle. By induction, one obtains an ordered sequence of positive cocycles $u = u_0, u_1, \cdots, u_N$ such that:

- For $1 \leq i \leq N$, u_i is obtained from u_{i-1} by a δ_{v_i}-move.
- All the crossings $v_1, \cdots, v_N$ are distinct.
- There is no crossing v distinct from $v_1, \cdots, v_N$ so that a non-negative δ_v-move can be applied to u_N.

One has then two cases:

1. The set $\{v_1, \cdots, v_N\}$ is the set of crossings of K.
2. There is a crossing w of K which does not belong to $\{v_1, \cdots, v_N\}$.

In the case (2), by the claim used in the proof of lemma 3.6, the cocycle u_N is zero on an incoming edge e at w.
This implies that u_N is also zero on some incoming edge of $K^{(1)}_{sing}$ at the crossing $i(e)$. Otherwise, a non-negative $\delta_{i(e)}$-move might be applied to u_N. This would imply that the crossing $i(e)$ is one of the crossings $\{v_1, \cdots, v_N\}$. Since a non-negative δ_v-move has been applied to each of this crossing to obtain the cocycle u_N, and no non-negative δ_w-move, $w = t(e)$, has been applied, this would imply that u_N is positive on e. This is a contradiction with our assumption.
By induction, as in lemma 3.6, one constructs a positive loop l in the singular graph such that $u_N(l) = 0$. Since u_N is cohomologous to u (see remark 2.3), this is a contradiction with u positive.

Therefore, case (1) above is satisfied. It remains to prove that $u_N = u$. For any edge e of the singular graph, exactly one $\delta_{t(e)}$-move has been applied, which lowers the value on e of the corresponding cocycle u_j ($j \leq N$) by 1, and exactly one $\delta_{i(e)}$-move has been applied, which increases the value on e of the corresponding cocycle u_k ($k \leq N$) by 1. The other δ_v-move do not change the value of the corresponding cocycles on e. Since this holds for any edge e, these remarks easily imply $u_N = u$. This completes the proof of corollary 3.7. $\diamondsuit$

Proposition 3.8 *Let K be a flat dynamical 2-complex which admits a positive cocycle $u \in C^1(K; \mathbf{Z})$.*
This cocycle defines a non-unique ordered sequence of $N+1$ graphs $\Gamma_0, \cdots, \Gamma_N$, where N is the number of crossings in K and such that:

1. *They are disjointly r-embedded in K in a non-degenerate way, and Γ_0 is the graph associated to u by lemma 2.9.*
2. *Each graph Γ_i is obtained from Γ_{i-1} by a Whitehead move for $i = 1, \cdots, N$ and Γ_N is homeomorphic to Γ_0.*

The graphs $\Gamma_0, \cdots, \Gamma_N$ are the leaves of a transversely oriented regular foliation $\mathcal{F}$ of K whose leaves are compact and such that:

- *Their transverse orientation agrees with the orientation of the edges of the singular graph.*
- *All the non-degenerate leaves of $\mathcal{F}$ are homeomorphic to one of the graphs $\Gamma_1, \cdots, \Gamma_N$ and there are exactly N degenerate leaves.*

Proof of proposition 3.8: Let us first assume that K has at least one crossing. From corollary 3.7, one has a sequence of positive cocycles $u = u_0 \to u_1 \to \cdots \to u_N = u$ such that u_i is obtained from u_{i-1} by a δ_{v_i}-move, and $\{v_1, \cdots, v_N\}$ is the set of crossings of K. Lemma 2.15 allows to obtain N disjoint subcomplexes $\mathcal{C}_{uu_1}, \mathcal{C}_{u_1u_2}, \cdots, \mathcal{C}_{u_{N-1}u_N}$ of K, which are homeomorphic to elementary foliated complexes $\mathcal{C}_{\Gamma_u\Gamma_{u_1}}, \mathcal{C}_{\Gamma_{u_1}\Gamma_{u_2}}, \cdots, \mathcal{C}_{\Gamma_{u_{N-1}}\Gamma_{u_N}}$. These subcomplexes contain all the crossings of K, and each one contains exactly one crossing. Therefore, the complement in K of $\mathcal{C}_{uu_1} \cup \mathcal{C}_{u_1u_2} \cup \cdots \cup \mathcal{C}_{u_{N-1}u_N}$ has N connected components $B_1, \cdots, B_N$. Each connected component B_i has two boundary components which are homeomorphic to Γ_{u_i}, $i = 1, \cdots, N$ and is foliated by $\Gamma_{u_i} \times [0,1]$. The union $\mathcal{C}_{uu_1} \cup (\Gamma_{u_1} \times [0,1]) \cup \mathcal{C}_{u_1u_2} \cup (\Gamma_{u_2} \times [0,1]) \cup \cdots \cup \mathcal{C}_{u_{N-1}u_N} \cup (\Gamma_{u_N} \times [0,1])$ gives a transversely oriented regular foliation with compact graphs which, by definition of an elementary foliated complex, satisfies all the properties given by proposition 3.8. If K has no crossing, the 2-components of K are either annuli or Moebius-bands (see the definition of a flat 2-complex and definition 1.2). The conclusion in this case is then straightforward, one just has to push a r-embedded graph Γ_u along the positive loops of the singular graph to obtain a foliation of K by graphs all homeomorphic to Γ_u. Proposition 3.8 is proved. $\diamondsuit$

Proposition 3.8 implies that if a flat dynamical 2-complex admits a positive cocycle, then there is a regular foliation $\mathcal{F}$, transversely oriented by the edges of the singular graph, whose leaves are compact and have all the same Euler characteristic. The equivalence of theorem 3.1 is thus proved.

3.3 Conclusion

One now completes the proof of theorem 3.1. Let Γ be a non-degenerate leaf of a foliation $\mathcal{F}$ as above. By cutting K along Γ, one obtains a 2-complex which is homotopically equivalent to $\Gamma \times [0,1]$. Thus the flat 2-complex K is homotopically equivalent to $Susp_\psi(\Gamma)$ (see section 1.1), where $\psi: \Gamma \to \Gamma$ is a continuous map which is a composition of maps induced by the sequence of Whitehead moves from Γ to Γ (see proposition 3.8). Since these maps are homotopy equivalences, the map ψ is an homotopy equivalence and thus it induces an automorphism on the fundamental group of Γ. This argument is easily generalized to the case where Γ is a degenerate leaf. The proof of theorem 3.1 is completed. $\diamondsuit$

Theorem 3.1 implies that a positive cocycle of a flat dynamical 2-complex K defines a continuous map of a trivalent graph which is r-embedded in K. This map appears as a composition of Whitehead moves. The proposition below gives a kind of converse to this result.

Proposition 3.9 *Let $\psi: \Gamma \to \Gamma$ be a continuous map of a trivalent graph Γ, such that $\psi = \alpha \circ \sigma_r \circ \cdots \circ \sigma_1$, is a composition of Whitehead-moves $\sigma_i: \Gamma_{i-1} \to \Gamma_i$ $(i = 1, \cdots, r)$, and of a homeomorphism $\alpha: \Gamma_r \to \Gamma_0$.*
Then there is a flat dynamical 2-complex K_S which is homotopically equivalent to the 2-complex $Susp_\psi(K)$. Moreover, K_S admits a positive cocycle u such that:

- *A graph Γ_u associated to u by lemma 2.9 is homeomorphic to Γ.*
- *Some ordered sequence of Whitehead-moves defined by u (see proposition 3.8) is the sequence given for the definition of ψ.*

Proof of proposition 3.9: Each Whitehead-move from Γ_{i-1} to Γ_i, $i = 1, \cdots, r$, is realized by a dynamic W-move (see definition 2.11). Each such move defines an elementary foliated 2-complex $\mathcal{C}_{\Gamma_{i-1}\Gamma_i}$ (see definition 2.13). The edges of its singular graph are oriented from Γ_{i-1} to Γ_i. One glues $\mathcal{C}_{\Gamma_{i-1}\Gamma_i}$ to $\mathcal{C}_{\Gamma_i\Gamma_{i+1}}$ by the identity of Γ_i, $i = 1, \cdots, r-1$. The 2-complex obtained, denoted by K', has two boundary components Γ_0 and Γ_r. One identifies Γ_r to Γ_0 by α. The 2-complex obtained is denoted by K_S.

One has to prove that K_S satisfies all the properties of a flat dynamical 2-complex. By lemma 2.14, an elementary foliated complex is a standard 2-complex with boundary. This easily implies that K' satisfies the same property. Therefore, since α is an homeomorphism, any point in K_S admits a neighborhood homeomorphic to a neighborhood of some point in the interior of the cone over the 1-skeleton of the boundary of the tetrahedron.

Let us now prove that the components of K_S are so that K_S is a flat 2-complex. Let e be any 1-component of Γ. We are first interested in the components of K'.
If e is not collapsed by any Whitehead move in the given sequence, then it gives rise to a $[0, 1] \times [0, 1]$ component in K'.
Otherwise, e gives rise to a finite set of 2-components $S_1, \cdots, S_k$ of K' such that, $\overline{S_i}$ denoting the closure in K' of S_i:

- For each $m = 1, \cdots, k-1$, $\overline{S_m} \cap \overline{S_{m+1}}$ is a crossing of K'.
- The component S_1 is a 2-cell which contains exactly one attractor in its boundary at $\overline{S_1} \cap \Gamma_1$ and $\overline{S_1} \cap \Gamma_0 = C$.
- For $p = 1$ to p such that $k = 2p$ or $k = 2p+1$, S_{2p} and S_{2p+1} are 2-cells of K'.
- The components $S_2, \cdots, S_{k-1}$ contain each exactly one attractor and one repellor in their boundary.

- The component S_k contains exactly one repellor in its boundary at $\overline{S}_k \cap \Gamma_{r-1}$ and $\overline{S}_k \cap \Gamma_r$ is a 2-cell.

Since K_S is the quotient of K' under $\alpha\colon \Gamma_r \to \Gamma_0$ and α is an homeomorphism, all the assertions above imply easily the following two properties:

- The 2-components of K_S are either 2-cells or are annuli or Moebius-band components.
- Each 2-component which is a 2-cell has exactly one attractor and one repellor in its boundary and the other components have none.

Thus K_S is a flat dynamical 2-complex. Moreover, by construction, the graph Γ_0 is r-embedded in K_S. The construction and the chosen orientation of the edges of the singular graph assure that, equipped with the good transverse orientation, it defines a positive cocycle and that this cocycle defines the given sequence of Whitehead moves. This completes the proof of proposition 3.9. $\diamondsuit$

The construction we use for proving the above proposition was suggested by G.Levitt. One presents another construction in [12].
The following corollary comes straightforward from proposition 3.9, using the well-known result that any free group automorphism can be expressed as a composition of Whitehead moves.

Corollary 3.10 *Let $\mathcal{O}$ be any automorphism of the free group F_n ($n \geq 1$). Then there is a flat dynamical 2-complex K which admits a positive cocycle $u \in C^1(K; \mathbf{Z})$ such that:*

- *The fundamental group of K is the suspension of the automorphism $\mathcal{O}$ of F_n.*
- *If Γ_u is as given by lemma 2.9, then any sequence of Whitehead moves defined by u on Γ_u induces the automorphism $\mathcal{O}$ on $\pi_1(\Gamma_u) \equiv F_n$, up to conjugacy in the group of outer automorphisms of F_n.*

In particular, from this corollary, any free group automorphism is represented by a flat dynamical 2-complex.

4 Non-singular semi-flows

In this section, we justify the name of *dynamical* 2-complex by proving that any such 2-complex carries a *non-singular semi-flow.*

Definition 4.1 A *non-singular semi-flow* on a topological space X is a family of continuous maps of X depending continuously on one parameter $t \in \mathbf{R}^+$ and such that:

- For all $x \in X$, $\sigma_0(x) = x$.
- For all t, t' in $\mathbf{R}^+$, $\sigma_{t+t'}(x) = \sigma_t(\sigma_{t'}(x))$.
- The set $\{x \in X$; for all t in $\mathbf{R}^+ \sigma_t(x) = x\}$ is empty.

If K is a flat 2-complex, one further requires that a non-singular semi-flow on K restricts to a C^∞ non-singular flow on each open 2-component of K (we assume them to be equipped with a structure of smooth manifold).
A graph Γ which is embedded and 2-sided in K is *transverse to the semi-flow* if all its intersection-points with the orbits of the semi-flow are transverse, and with the same intersection-sign, once a transverse orientation to Γ has been chosen.
A *cross-section* to a non-singular semi-flow on a flat 2-complex K is a transverse graph Γ which is r-embedded in K and which intersects all the orbits in finite time.
As in the usual case of a flow on a manifold, when a non-singular semi-flow admits a cross-section, there is a well-defined continuous return-map of the semi-flow on the cross-section.

In what follows, we will assume that a graph transverse to a semi-flow is transversely oriented by the semi-flow.

4.1 Combinatorial semi-flows

In what follows, the *triangle* T denotes the cone, based at the origin $(0,0)$ of the oriented plane $\mathbf{R}^2$, over the interval $y = 1-x$, $x \in [0,1]$. We denote by R the square $[0,1] \times [0,1]$.
The *model-flow on* T (resp. on R) is the restriction to T (resp. to R) of the non-singular flow on $\mathbf{R}^2$ whose orbits are the lines $y = \mu - x$, $\mu \in [0,1]$ (resp. the lines $x = \mu$, $\mu \in [0,1]$).

Definition 4.2 A *combinatorial semi-flow* on a dynamical 2-complex K is a non-singular semi-flow on K satisfying the following properties:

1. There is a decomposition of K in a finite number of triangular and rectangular boxes whose boundary-points are pre-periodic under the semi-flow and such that the semi-flow in restriction to each box is topologically conjugate to the corresponding model-flow.

2. The orientation of the semi-flow agrees, in a neighborhood of the singular graph $K^{(1)}_{sing}$, with the orientation of the edges of $K^{(1)}_{sing}$.

3. In each disc component, an orbit-segment connects the repellor to the attractor.

4. Let X be a 2-component which is either a coherent annulus component or a Moebius-band. Then the semi-flow is transverse to the rays of X and the core of X is a periodic orbit.

Proposition 4.3 *Any flat dynamical* 2*-complex* K *carries a combinatorial semi-flow.*

Before beginning the proof, we need to introduce some terminology. One chooses an embedding in $\mathbf{R}^3$ of a neighborhood $N(v)$ in K of each crossing v, which induces an embedding in the horizontal plane $\mathbf{R}^2$ of $K^{(1)}_{sing} \cap N(v)$. The singular set $K^{(1)}_{sing} \cap N(v)$ decomposes then in a unique way as the union of two intervals, oriented by the edges of the singular graph, which intersect transversely. These intervals are identified to the oriented axis of the horizontal plane. All the germs of 2-components at v are assumed to lie in a coordinate-plane, corresponding to the germs of edges that they contain. Let e be any edge of the singular graph incident to a crossing v. We write e as the concatenation of two positive intervals e_1e_2. A germ of 2-component at v lies *on the* 1*-side of* e *at* $i(e) = i(e_1)$ (resp. *at* $t(e) = t(e_2)$) if this germ of 2-component contains a germ g of e_1 (resp. of e_2) at $i(e)$ (resp. at $t(e)$) and also contains a germ of outgoing edge at $i(e)$ (resp. at $t(e)$) which does not lie in the same axis than g. The germs of 2-component at $i(e)$ satisfying the first property but not the second lie *on the* 2*-side of* e *at* $i(e)$ (resp. *at* $t(e)$).

Proof of proposition 4.3: One assumes that a collection of embeddings in $\mathbf{R}^3$ of small neighborhoods of the crossing, pairwise disjoint, has been chosen as above. In each such neighborhood, one defines a non-singular semi-flow transverse to the singular set, going from the 2-side of each edge to its 1-side, in such a way that it satisfies at each point property (2) of definition 4.2 (see figure 6). These semi-flows are called below *crossing semi-flows.*

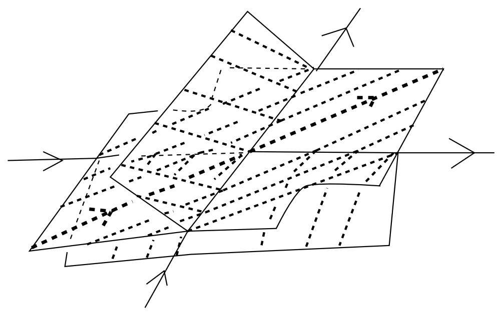


Figure 6: A crossing semi-flow

Let e be any edge of the singular graph. Let us recall that the neighborhood of a point x interior to e is homeomorphic to $T_x \times [0, 1]$, where T_x is a triod centered at x (see lemma 1.1, item (1)).

1. If the three germs of 2-cells at e are on the same side of e at $i(e)$ and $t(e)$, then we extend the crossing semi-flow through the neighborhood of e in such a way that it is everywhere transverse to e and has the same direction at every point of the neighborhood.

2. Otherwise, there are exactly two germs of 2-components which change side from $i(e)$ to $t(e)$. Then we extend the semi-flow through the neighborhood of e in the following way:

   - It has the same direction at each point of the germ of 2-component which does not change side from $i(e)$ to $t(e)$.
   - The direction of the semi-flow at the points in e varies continuously from $i(e)$ to $t(e)$.
   - Let x be a point interior to e and let T_x be the triod over x. The direction of the semi-flow along the two arms of T_x which belong to the 2-components changing side from $i(e)$ to $t(e)$ is the same than its direction at x.

By applying this construction along each edge of the singular graph, one obtains a non-singular semi-flow in a neighborhood $\mathcal{N}(K^{(1)}_{sing})$ of the singular graph, whose

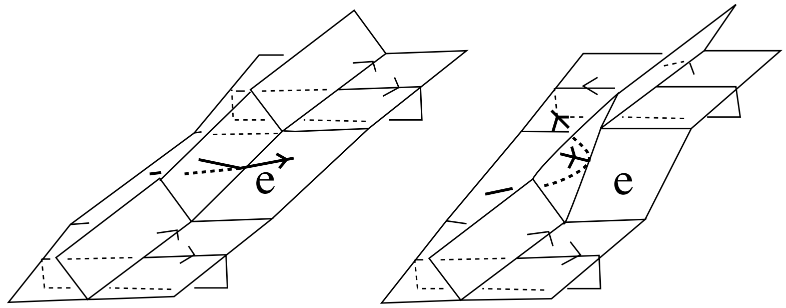


Figure 7: Extending a semi-flow through an edge

orientation agrees with the orientation of the edges. The point now is to extend this semi-flow on $\mathcal{N}(K^{(1)}_{sing})$ to a semi-flow on K without adding fixed points.

Let D be any disc in the complement in K of $\mathcal{N}(K^{(1)}_{sing})$. Since K is a dynamical 2-complex, the disc D has exactly one repellor R and one attractor A in its boundary. The boundary ∂D, cut at R and A, decomposes in two positive paths p_1, p_2 oriented from R to A (see lemma 1.3). By construction, there are no singularities of the semi-flow along these paths, and the orientation of the semi-flow agrees with their respective orientation. With respect to D, the semi-flow is incoming (resp. outgoing) in a neighborhood of R (resp. A). We denote by k_i the number of external tangencies of the semi-flow with the path p_i. Let us observe that there are at least two points of external tangency, and between two such points, there is a point of internal tangency.

We now define the *vertical flow* on $\mathbf{R}^2$ to be the flow whose trajectories are the lines $x = C$, $C \in \mathbf{R}$, oriented from $-\infty$ to $+\infty$. One draws a curve in $\mathbf{R}^2$ which is modelled on $y = x^2$ (resp. $y = -x^2$) at $(0,0)$ (resp. $(0,1)$), on $x = y^2$ (resp. $x = -y^2$) at k_1 points $(-2, \frac{1}{k_1+1}), \cdots, (-2, \frac{k_1}{k_1+1})$ (resp. at k_2 points $(2, \frac{1}{k_2+1}), \cdots, (2, \frac{k_2}{k_2+1})$), on $x = -y^2$ (resp. $x = y^2$) at $k_1 - 1$ points $(-1, \frac{3}{2(k_1+1)}), \cdots, (-1, \frac{k_1+1}{2(k_1+1)})$ (resp. at $k_2 - 1$ points $(1, \frac{3}{2(k_2+1)}), \cdots, (1, \frac{k_2+1}{2(k_2+1)})$). Therefore, the curve is orthogonal to the vertical flow at $(0,0)$ and $(0,1)$, tangent to this flow at all the other points defined above. It is easy to check that it can be drawed tranverse, and non-orthogonal, to this flow outside these points. See figure 8.

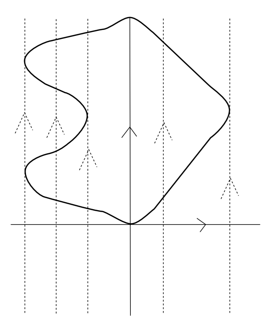


Figure 8: Extending a semi-flow though a disc component

This curve bounds a disc $\mathbf{D}$. By construction, the vertical flow has the same behaviour along $\partial\mathbf{D}$ than the semi-flow along ∂D. Thus, one can choose a diffeomorphism from $\mathbf{D}$ to D which carries the restriction of the vertical flow on $\mathbf{D}$ to a non-singular flow on D in such a way that it extends the semi-flow already defined along the boundary curve ∂D.

The construction for extending the semi-flow through the coherent annulus and through the Moebius-bands in the complement in K of $\mathcal{N}(K^{(1)}_{sing})$ is similar. One just substitutes horizontal intervals to the attractors and repellors. Let us now consider an incoherent annulus in the complement of $\mathcal{N}(K^{(1)}_{sing})$. One defines two periodic orbits in its interior, which are oppositely oriented and such that, by cutting the annulus along them, one obtains three annuli, two of which are coherent. For extending the semi-flow through these two annuli, the construction is the same than above. In the last annulus, one just defines the semi-flow to form a Reeb-component (see figure 9).

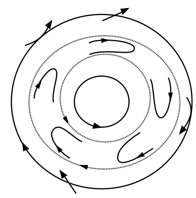


Figure 9: A Reeb component in an annulus

One so obtains a non-singular semi-flow on K. After possibly some small pertubations, one obtains a non-singular semi-flow such that the points of tangency with the boundary of the components are pre-periodic, and the cores of the coherent annulus components, and of the Moebius-band components are periodic. It is then an easy task to obtain a combinatorial semi-flow. This completes the proof of proposition 4.3. $\diamondsuit$

Proposition 4.4 *If a flat dynamical* 2*-complex* K *admits a positive cocycle* $u \in C^1(K; \mathbf{Z})$*, then any combinatorial semi-flow* $(\sigma_t)_{t \in \mathbf{R}^+}$ *on* K *admits some r-embedded graph* Γ_u *as given by lemma 2.9 as a cross-section.*

Proof of proposition 4.4: By definition of a combinatorial semi-flow, in each disc component D, an orbit-segment of $(\sigma_t)_{t \in \mathbf{R}^+}$ connects the repellor R to the attractor A. By lemma 1.3, it cuts D in two connected components whose boundary is the union of this orbit-segment with a positive path p_i, $i = 1, 2$, in the singular graph $\mathcal{S}$ of K, going from R to A.

Lemma 4.5 *Let* D *be either a disc component, or a coherent annulus or Moebius-band component. Then any orbit-segment of* $(\sigma_t)_{t \in \mathbf{R}^+}$ *in* D *which is transverse to* ∂D *at its endpoints (one will say* properly embedded*) is homotopic, relative to these endpoints, to a positive path in* $\mathcal{S}$*.*

Proof of lemma 4.5: Assume that this property is not satisfied by some properly embedded orbit-segment s. The existence of the orbit-segment connecting R to A implies that the endpoints of s in ∂D belong to a same positive path p_i. The above assumption implies that the incoming endpoint $i(s)$ of s follows, along p_i, the outgoing one $t(s)$. Let p'_i be the positive subpath of p_i from $t(s)$ to $i(s)$. The union of s and p'_i forms an oriented loop, whose orientation agrees with the orientation of the semi-flow (see figure 10). This loop bounds a disc in D. This implies that there is a singularity of the semi-flow in the disc. This is impossible. Therefore, any properly embedded orbit-segment of $(\sigma_t)_{t \in \mathbf{R}^+}$ in D is homotopic relative to its endpoints to a positive path in $\mathcal{S}$.
The same conclusion than above remains for the orbit-segments in the coherent annulus components, or in the Moebius-band components. The core of the component,

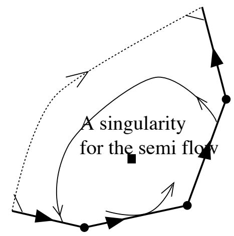


Figure 10: All orbits are positive paths.

which by definition of a combinatorial semi-flow is a periodic orbit, plays the role of the orbit-segment connecting the repellor to the attractor. $\diamondsuit$

Since K admits a positive cocycle, K does not have any incoherent annulus component. Since $\mathcal{S}$ is finite, and since, by definition of a positive cocycle, any r-embedded graph Γ_u intersects positively all the positive loops in $\mathcal{S}$, lemma 4.5 gives the following corollary:

Corollary 4.6 *For any point x in K, the algebraic intersection-number of some finite union of properly embedded orbit-segments of $(\sigma_t)_{t\in\mathbf{R}^+}$ through x with any r-embedded graph Γ_u is strictly greater than one.*

For proving that some r-embedded graph Γ_u is a cross-section to $(\sigma_t)_{t\in\mathbf{R}^+}$, it remains to prove that some such Γ_u is transverse to $(\sigma_t)_{t\in\mathbf{R}^+}$.

Lemma 4.7 *Let K be a flat dynamical 2-complex. Let $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$ be any combinatorial semi-flow on K. If x, y are any two points in the boundary of a disc component D, which lie on both sides of a line connecting the repellor to the attractor, there exists a segment in D transverse to $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$ which connects x to y. The same assertion is true if D is a coherent annulus component, or a Moebius-band component, and x and y are on both sides of the core of D.*

Proof of lemma 4.7: Choose a segment u connecting x to y, and transverse to the orbit-segment O_{RA} of $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$ connecting the repellor R to the attractor A of D. If u is transverse to $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$, one is done. Let us assume therefore that there exists at least one point of tangency T_0 between u and $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$. Without loss of generality, one can assume that these points of tangency are isolated and in finite number.
One considers now a small rectangular neighborhood N_0 of T_0 in K. One slightly pushes the segment $u\cap N_0$ along the normal to $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$ in N_0. If the tangency point does not disappear, i.e. one has a sequence of quadratic contacts with the semi-flow (see figure 11), then one obtains a new segment u_1 in the horizontal boundary of N_0, which contains a point of tangency T_1 with $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$. One considers then a new small, rectangular neighborhood N_1 of T_1, overlapping with N_0, and executes the same process. By induction, one so constructs a chain of overlapping small rectangular boxes N_i. One stops either when one has to pass through the orbit-segment O_{RA}, or when the horizontal boundary of N_i meets ∂D, or when the tangency point disappears. Let us observe that, at the first step, one has the choice to push in one direction or in the other along the normal to the semi-flow. If, once chosen one direction, after some steps one cannot push further in this direction because one met ∂D, then one returns to the initial point of tangency T_0 and pushes in the other direction. Clearly, the chain so constructed is finite.

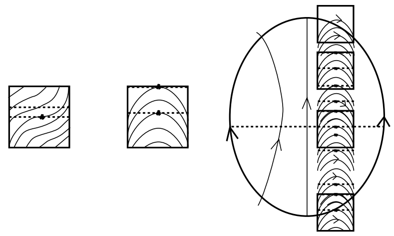


Figure 11: Pushing a segment along a normal to $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$

The important point now is that, since the orientation of $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$ agrees, in a neighborhood of the singular graph, with the orientation of the edges, this chain does not connect the boundary of D to itself (see figure 11). Furthermore, from lemma 4.5, no properly embedded orbit-segment of $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$ is homotopic, relative to its endpoints, to a negative path in the singular graph. From these two observations, constructing a new segment connecting x to y with one less tangency point with $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$ than u is now an easy task. We refer the reader to figure 12: the first picture illustrates the case where pushing along the normal to $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$ makes the intersection-point disappear, the second picture corresponds to the case where one meets the orbit-segment O_{RA}.

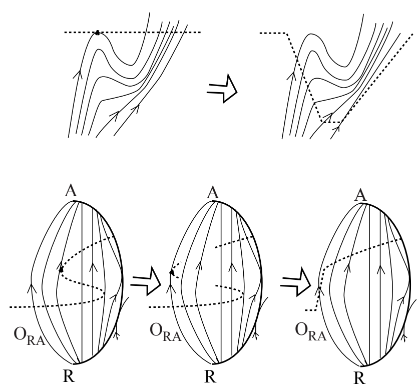


Figure 12: Eliminating points of tangency

This allows to eventually obtain an arc connecting x to y and transverse to the semi-flow in D. The arguments are the same if D is a coherent annulus component, or a Moebius-band component. $\diamondsuit$

Lemma 4.7 allows to get a r-embedded graph Γ_u transverse to the given combinatorial semi-flow. Together with which precedes, this completes the proof of proposition 4.4. $\diamondsuit$

Remark 4.8 Let K be a flat dynamical 2-complex.

1. There exists a regular foliation $\mathcal{F}$ of K transversely oriented by the edges of the singular graph if and only if there are no incoherent annulus components. In particular, any standard dynamical 2-complex admits such a foliation.

2. If K admits a regular foliation $\mathcal{F}$ transversely oriented by the edges of the singular graph, then any combinatorial semi-flow on K is transverse to such a foliation. In particular, any combinatorial semi-flow on a standard dynamical 2-complex is transverse to such a foliation.

5 Remarks

5.1 Dynamical 2-complexes and branched surfaces

In what follows, we discuss the relationships between the dynamical 2-complexes and other classes of 2-complexes already existing in the literature, which are more or less close to the dynamical 2-complexes introduced here.

A *branched surface* W is a 2-complex equipped with a smooth structure such that a tangent plane T_xW, varying continuously in x, is defined at each point x of the complex. Branched surfaces have known a great success, since their first apparition in the work of Williams in 1973 ([24]), in several areas: In dynamical systems, they have been used, also under the name of *templates*, as well for the study of Lorenz attractors (see [3, 4, 25] for instance) as for the study of flows on 3-dimensional manifolds, for instance through the knots formed by their periodic orbits (see [3, 4, 15]). Under another form, branched surfaces also appeared more recently in the work of Christy (see [6, 7, 8]) or Benedetti-Petronio (see [2]).

The first defines the *dynamic branched surfaces*, which are branched surfaces with the property to carry a non-singular semi-flow. Topologically, a dynamic branched surface is like a dynamical 2-complex. The existence of this tangent plane allows to distinguish, at each point of the complex, a locally 2-sheeted and a locally 1-sheeted side. At the difference of a dynamical 2-complex, the semi-flow on the dynamic branched surface is transverse to its singular graph, going from the locally 2-sheeted to the locally 1-sheeted side. Not any dynamical 2-complex is a dynamic branched surface and not any dynamic branched surface is a dynamical 2-complex. One can define in a natural way a smoothing in a neighborhood of the crossings of a dynamical 2-complex by requiring that the locally 2-sheeted side is the 2-side of each edge in these neighborhoods. However, such a smoothing does not necessarily extend through the edges of the singular graph, because some germ of 2-component might change side along some edge (see figure 7). Conversely, one can define in a natural way an orientation on the germs of edges of $W^{(1)}_{sing}$ at the crossings of a dynamic branched surface W. However, these orientations do not necessarily define an orientation of the edges of the singular graph (see figure 13).

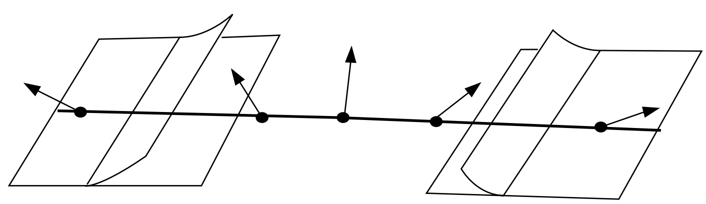


Figure 13: A singularity for the orientation induced by a semi-flow

This figure illustrates an essential difference between dynamic branched surfaces and dynamical 2-complexes, that is the edges of the singular graph of a dynamic branched surface are not necessarily coherently oriented by the semi-flow. There are some points in the interior of the edges of $W^{(1)}_{sing}$ at which the semi-flow is orthogonal to the edge. J.Christy introduced these dynamic branched surfaces for the study of hyperbolic attractors in dimension 3.

Benedetti and Petronio introduced the *branched standard spines*. A non-singular flow is also present here, transverse to the spine. These branched spines thus give in some sense a combinatorial coding of a flow. The authors give then a caracterization of the homotopy-class of this flow, by proving that two branched standard spines define homotopic flows if and only if one can pass from one to the other by a finite number of well-defined moves on the spines. At this point, two remarks are to be done: On one hand, the authors do not make any reference to the hyperbolicity of their flows. This would perhaps be an interesting exercise to be able to determine at which condition a flow presented by a branched spine is an Anosov, or pseudo-Anosov, flow. On the other hand, neither is it true that a branched standard spine is a dynamic branched branched surface, nor the converse. Indeed, whereas the Euler characteristic of a dynamic branched surface vanishes, which is not necessarily the case for a branched standard spine, this last one is embedded in a compact 3-manifold and tranversely oriented, which is not necessarily the case for a dynamic branched surface.

In parallel to these developments in dynamical systems, branched surfaces played also an important role in 3-dimensional topology, starting from the works of Oertel on incompressible surfaces (see [9, 19] for instance) and continuing with the works of Gabai, among others, about laminations (see [11] for instance). Let us observe that, anterior to the branched surfaces, in topology appeared the standard spines in the work of Casler ([5]). The generic branched surfaces defined by Williams in [24] in particular were standard spines. However, both objects evolved in different worlds until very recently, and J.Christy, in [6], was the first to establish a link between them.

5.2 Connectedness of the leaves

From our assumption on the connectedness of the CW-complexes considered in this paper, when one is given a leaf of a foliation, this leaf is assumed to be connected. However, when one is given a positive cocycle u of a dynamical 2-complex K, an associated r-embedded graph K_u (see lemma 2.9) is not necessarily connected. The fact that u defines a regular foliation with compact leaves of K allows to prove that a positive cocycle u defines a connected r-embedded graph K_u if and only if the cohomology class of u is *indivisible*, that is there exists a loop l in K with $u(l) = 1$.

5.3 Mapping-tori of surface homeomorphisms

In the context of *spines* of 3-manifolds (see [1], [5], [17] for instance), any non-singular semi-flow $\{(\sigma_t)\}_{t\in\mathbf{R}^+}$ on a *dynamical* 2-*spine* (that is a dynamical 2-complex which is a spine of some compact 3-manifold with boundary) will give rise to a non-singular semi-flow on the ambient manifold. We refer the reader to [13] for more details. However, a nice way for the construction of non-singular flows on 3-manifolds, starting from a standard dynamical 2-spine K, is to consider K as the 2-skeleton of the dual to a union of tetrahedra $\mathcal{T}$, glued together along their faces (see figure 14).

More precisely, each crossing of K is the barycenter of a tetrahedron, the edges of $K^{(1)}_{sing}$ are dual to the faces, each 2-component is dual to an edge of $\mathcal{T}$. The link of each vertex, in the second barycentric subdivision of $\mathcal{T}$, is deleted. One so obtains an embedding of K in an oriented compact 3-manifold with boundary M_K whose it is a spine. Now, the orientation of the edges of the singular graph allows to define a non-singular flow through each tetrahedron, the flow being incoming through the faces dual to the incoming edges at the corresponding crossing, and outgoing through the two other faces. The gluing of the tetrahedra defines then a non-singular flow on M_K.

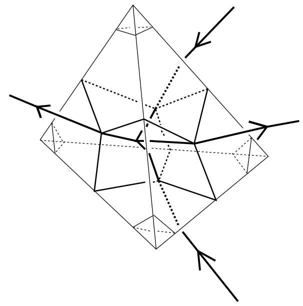


Figure 14: Embedding a standard 2-complex in a 3-manifold

Furthermore, in this topological context, one can prove the following topological analog to corollary 3.10 of proposition 3.9:

Any compact 3*-manifold with boundary* M^3 *which admits a fibration* f *over the circle admits a dynamical* 2*-spine* K *with a positive cocycle* $u \in C^1(K;\mathbf{Z})$ *such that* $i_\#([u])$ *is in the cohomology class of* $H^1(M^3;\mathbf{Z})$ *defined by* f*, where* $[u]$ *denotes the cohomology-class of* u *in* $H^1(K;\mathbf{Z})$ *and* $i_\#\colon H^1(K;\mathbf{Z}) \to H^1(M^3;\mathbf{Z})$ *the isomorphism induced by the inclusion* i *of* K *in* M^3*.*

Indeed, any such 3-manifold is the mapping-torus of a homeomorphism h of a compact surface with boundary S. Up to isotopy, this homeomorphism can be given by a composition of Whitehead moves and of a homeomorphism ϕ applied on a spine Γ of S, in such a way that each Whitehead move and the homeomorphism ϕ preserve the embedding in S. From such a decomposition, one easily checks that proposition 3.9 gives a dynamical 2-spine K of M^3. The properties of K listed in proposition 3.9 assure that K is as announced above.

5.4 Effectivity of the cocycle-criterion

The criterion given by theorem 3.1 for the existence of a foliation with compact leaves of a flat 2-complex is effective, that is one can easily check, by hand or by a computer, if a given flat 2-complex admits a positive cocycle. Let us briefly describe this process. Assume that you are given some flat 2-complex K. Check if the edges of its singular graph admit an orientation which makes it a dynamical 2-complex. Since the singular graph is finite, this is a finite process. In a second step, consider the integral matrix whose lines are the images of the 2-components of the complex by the second boundary-operator, and the columns are the edges of the singular graph. Supress from this matrix the lines corresponding to the Moebius-band components. Let us denote by M_K the resulting matrix. Search for the non-negative integer solutions to the system $M_K X = 0$. A classical result asserts that these non-negative solutions are generated by a finite number of them, i.e. there are n such solutions $S_1, \cdots, S_n$ of this system such that, if S is any non-negative integer solution, then $S = \sum_{i=1}^{n} \lambda_i S_i$, $\lambda_i \geq 0$. Each solution S_i does not necessarily define a cocycle of the 2-complex K, but, in any case, one easily shows that $2 * S_i$ will define a cocycle. Thus, for testing if there is a positive cocycle, it suffices to consider the sum $2 * \sum_{i=1}^{n} S_i$. There is a positive cocycle in $C^1(K;\mathbf{Z})$ if and only if this sum is one.
Let us observe that all what preceeds implies in particular that, on a given flat dy-

namical 2-complex K, there might be an infinite number of indivisible non-negative cohomology classes (it is necessary that $rk(H_1(K;\mathbf{Z})) > 1$).

Examples

Example 1: In this example, we assume given a trivalent graph with two vertices, together with a cyclic ordering at each of these vertices. We consider a sequence of two Whitehead moves from this graph to an homeomorphic one. Moreover, the two Whitehead moves and the homeomorphism preserve the above cyclic orderings. Up to homeomorphism, there is a unique orientable compact surface with boundary which admits this graph, *equipped with these cyclic orderings at its vertices*, as a spine. We leave the reader check that this surface S is the torus with one boundary component. The above sequence of Whitehead moves and homeomorphism defines a continuous map of the graph, induced by a homeomorphism h of S. In fact, this homeomorphism is induced by the classical automorphism $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ of the torus.
We apply proposition 3.9 for constructing a suspended dynamical 2-complex of the induced automorphism on the fundamental group. Since the Whitehead moves and the homeomorphism preserve the cyclic orderings of the edges at the vertices, one gets a flat dynamical 2-spine of the 3-manifold which is the suspension of the homeomorphism h of S. This manifold is the complement in $\mathbf{S}^3$ of the figure eight-knot.
Let us describe this construction. The edges of the graph are labelled with 1, 2 and 3. When one edge is collapsed by a Whitehead move, the symbol attached to the new edge created is the same than the original one, with a prime. The homeomorphism α is defined by $\alpha(2') = 3$, $\alpha(3') = 1$ and $\alpha(1) = 2$.
Since there are two Whitehead moves, the singular graph of the suspended dynamical 2-complex is a 4-valent graph with two crossings.
The second picture in figure 15 illustrates what happens to each edge along the process, the graph Γ having been cut at its vertices. It allows to find the 2-components of the complex, together with a decomposition of their boundary in 1-simplices.
Since the edge 1 is collapsed by no Whitehead move, it gives rise to a rectangle. The two other edges are collapsed. Thus, each of them gives rise to two triangles: these are the cones over the edges 2, $2' = \alpha^{-1}(3)$ and 3, $3' = \alpha^{-1}(1)$. The vertex common to the two triangles with bases 2 and $2'$ is a crossing of the complex, and the same is true for the vertex common to the two triangles with bases 3 and $3'$. For obtaining the desired suspended 2-complex, one identifies the top and the bottom of two "bands" when this top and bottom carry the same letters.
The suspended 2-complex has two 2-components. The boundary of each 2-cell inherits a decomposition in 1-simplices which are copies of the edges of the singular graph along which this 2-cell is attached. For finding this decomposition of the boundary of each 2-cell, let us notice that one took care of not putting the two crossings at the same level in this second picture of figure 15. Since all the edges of Γ are incident to both vertices of Γ, it suffices then to subdivide the vertical boundaries of the "bands" at the level of the crossings to obtain the desired decomposition of the boundary of the 2-cells in 1-simplices. These 1-simplices, copies of the edges of the singular graph, are oriented from top to bottom in this picture.

The dynamical 2-complex K is showed in the third picture of figure 15. We also drawed a r-embedding of our original graph Γ into K. The interested reader can easily check that the associated cocycle is a positive one. Let us notice that K admits a compatible structure of dynamic branched surface (see [13]).
This dynamic branched surface first appeared in [6]. It is dual to the decomposition in two tetrahedra of the complement in $\mathbf{S}^3$ of the figure-eight knot given by Thurston in its notes (see [22]).

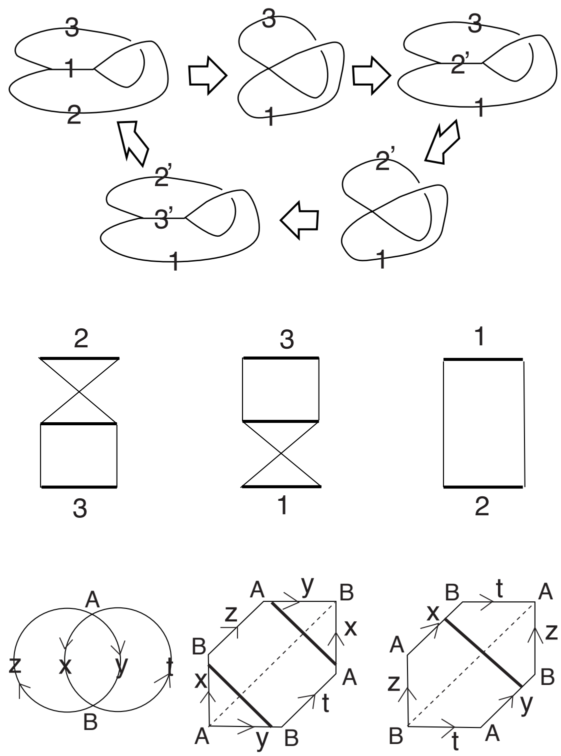
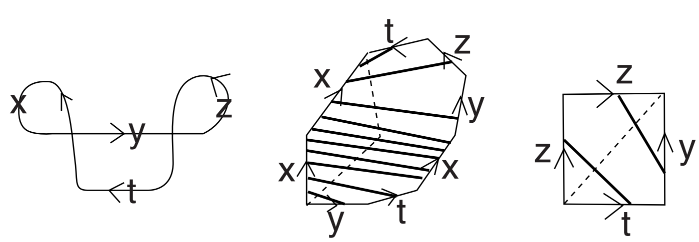


Figure 15: Example 1

Example 2: Figure 16 presents a flat dynamical 2-complex which admits a compatible structure of dynamic branched surface (see [13]). This complex admits a positive cocycle and we show a r-embedded graph associated to such a cocycle. One easily checks, using criterions of embeddability of Christy or Benedetti-Petronio (see [7, 1]), that this complex does not embed in any compact 3-manifold.



x = 5, y = z = t = 1

Figure 16: Example 2

Example 3: Figure 17 gives a dynamical 2-complex which does not admit any positive cocycle. There are only non negative cocycles which are not positive. For instance, the solution $X_2 = 2$, $X_7 = 2$, $X_4 = X_9 = 1$ gives such a cocycle. It remains, in the complement of the edges with positive weight, the positive loop $X_3X_{10}X_5$. As the example 2, this 2-complex does not admit any embedding in a compact 3-manifold. Let us observe that if we give to all the edges of the singular graph the opposite orientation, then the dynamical complex admits a compatible structure of dynamic branched surface (see [13]).

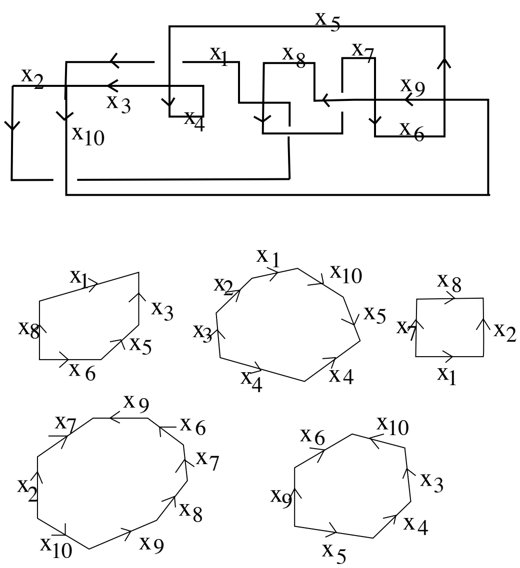


Figure 17: Example 3

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AMS classification: 57Mxx,20Fxx,57N10,54H20